On evaluation of two-loop self-energy diagram with three propagators

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Abstract

Small momentum expansion of the "sunset" diagram with three different masses is obtained. Coefficients at powers of p^2 are evaluated explicitly in terms of dilogarithms and elementary functions. Also some power expansions of "sunset" diagram in terms of different sets of variables are given.

The "sunset" diagram (see Fig. 1) was the object of investigation in several recent works [1, 2, 3, 4, 5]. Such activity is initiated, certainly, by rather high precision of experimental testing of the standard model that needs sometimes evaluation of two-loop Feynman diagrams for comparison of the theory and experiments.
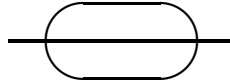

Figure 1: The "sunset" diagram

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Probably, the "sunset" diagram cannot be expressed through known special functions of one variable. So for investigation of this diagram it was used numerical methods [4, 5] and some approximations schemes, such as expansions in powers of p^2 (and possibly logarithms) [1, 2, 3] [1].

In particular, in paper [3] it was obtained small momentum expansion of "sunset" diagram in terms of Lauricella series. These series are convergent in the region

$$\sqrt{p^2} + m_1 + m_2 < m_3 \tag{1}$$

where m_i, $i = 1, 2, 3$, are masses in propogators. But for majority of applications $m_1 + m_2 > m_3$ for any numbering of masses. Such situation takes place, for instance, in the course of evaluation of self-energy of Higgs boson (when $m_1 = m_2 = m_3 = m_H$), and in the course of evaluation of contribution to Z-boson self-energy due to vertex Z^2WW^* (when $m_1 = m_2 = m_W$, $m_3 = m_Z$).

The aim of the present paper is to complete the results of [3] and to obtain small momentum expansion of "sunset " diagram that can be used for all values of masses, and to evaluate in closed form coefficients at powers of p^2. In addition, we will give some power expansions of "sunset" diagram that are convergent, at least, in the region

$$|p^2| + m_3^2 < (m_1 + m_2)^2 \tag{2}$$

Obviously, these expansions can be used for all values of masses if one defines m_3 as $\min\{m_1, m_2, m_3\}$.

Let $I = I(p^2, m_1^2, m_2^2, m_3^2)$ be the "sunset" diagram. Our main tool will be the following one-fold integral representation for the function I:

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[1] See also an old paper by Mendels [6] where it was obtained expansion of "sunset" diagram in equal mass case in powers of $p^2/p^2 + m^2$

$$I = \int_{(m_1+m_2)^2}^{\infty} d\sigma^2 \rho(\sigma^2, m_1, m_2) \left\{ J(p^2, m_3^2, \sigma^2) \right.$$

$$\left. -\pi^2 \left[\frac{m_3^2}{\sigma^2} \log \frac{m_3^2}{\sigma^2} + m_3^2 f_1(\sigma^2) + p^2 f_2(\sigma^2) + f_3(\sigma^2) \right] \right\} \qquad (3)$$

where J is one-loop "bubble" diagram (see Fig.2), ρ is spectral function of J,

$$\rho = \pi^2 \sqrt{\left(1 - \frac{(m_1 + m_2)^2}{\sigma^2} \right) \left(1 - \frac{(m_1 - m_2)^2}{\sigma^2} \right)} \qquad (4)$$

and functions $f_i, \quad i = 1, 2, 3$ must be defined in such a way that integral (3) converges.

Figure 2: The "bubble" diagram

Representation (3) was obtained in a previous author's work [7] [2]. Very similar integral representation was obtained independently in above mentioned papers [3, 4]. Another one-fold integral representation was given in [5] .An analogous one-fold integral representations for the five propogator self-energy diagram was derived in [8, 9].

One notes that, due to renormalization freedom, function I is defined up to polynomial of the first degree with respect to p^2. So it is sufficient to evaluate the function

$$\bar{I}(p^2, ...) = I(p^2, ...) - I(0, ...) - p^2 \frac{dI}{dp^2}(0, ...) \qquad (5)$$

Comparing (3) and (5), one obtains the following integral representation for $\bar{I}$:

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[2] In [7] representation (3) was derived only for the case $m_1 = m_2$, but, in fact, this condition was never used and for general case representation (3) can be obtained in the same way.

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$$\bar{I} = \int_{(m_1+m_2)^2}^{\infty} d\sigma^2 \rho(\sigma^2, m_1, m_2) \bar{J}(p^2, \sigma^2, m_3^2) \tag{6}$$

where

$$\bar{J}(p^2, ...) = J(p^2, ...) - J(0, ...) - p^2 \frac{dJ}{dp^2}(0, ...) \tag{7}$$

The function $\bar{J}$ can be represented by the following power series:

$$\bar{J} = \pi^2 \sum_{k \geq 2, l \geq 0} \frac{(k+l-1)!(k+l)!}{l!(2k+l+1)!} \left(-\frac{p^2}{\sigma^2}\right)^k \left(1 - \frac{m_3^2}{\sigma^2}\right)^l \tag{8}$$

The proof of (8) is very simple [3]. One considers the representation of J through Feynman parameters:

$$J = -\pi^2 \int_0^1 d\xi \log \left[\frac{p^2 \xi(1-\xi) + \xi m_3^2 + (1-\xi)\sigma^2}{\sigma^2}\right] \tag{9}$$

Expanding the integrand in powers series with respect to

$$\frac{p^2 \xi(1-\xi) + \xi(m_3^2 - \sigma^2)}{\sigma^2}$$

and integrating, one obtains (8). These operations are correct if

$$\max_{0 \leq \xi \leq 1} \left|\frac{p^2 \xi(1-\xi) + \xi(m_3^2 - \sigma^2)}{\sigma^2}\right| \leq 1 \tag{10}$$

From (10) one can derive that series (8) converges, at least, if

$$|p^2| + m_3^2 < \sigma^2 \tag{11}$$

One introduces dimensionless variables

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[3] Another proof was given in [10].

$$x = \frac{(m_1 + m_2)^2}{\sigma^2}, \quad y = \frac{m_3^2}{(m_1 + m_2)^2} < 1,$$

$$z = -\frac{p^2}{(m_1 + m_2)^2}, \quad \lambda = \frac{(m_1 - m_2)^2}{(m_1 + m_2)^2} \tag{12}$$

In these variables, using definition of Gauss hypergeometric function $_2F_1$, one can rewrite (8) in the following way:

$$\bar{J} = \pi^2 \sum_{k \geq 2} x^k z^k \frac{(k-1)!k!}{(2k+1)!} \, _2F_1(k, \ k+1; \ 2k+2; \ 1-xy) \tag{13}$$

But

$$\frac{(k-1)!k!}{(2k+1)!} \, _2F_1(k, \ k+1; \ 2k+2; \ u) = \frac{1}{k!(k+1)!} \frac{d^k}{du^k}(1-u)^{k+1} \frac{d^k}{du^k} \frac{\log(1-u)}{u} \tag{14}$$

(see, for instance, [11]). From the last formula, after some elementary transformations, one can obtain:

$$\frac{(k-1)!k!}{(2k+1)!} \, _2F_1(k, \ k+1; \ 2k+2; \ 1-xy)$$
$$= \frac{1}{k!(k+1)!} \frac{1}{x^{k-2}} \frac{d}{dx} \frac{d^k}{dy^k} y^{k+1} \frac{d^{k-1}}{dy^{k-1}} \frac{\log(xy)}{y(1-xy)} \tag{15}$$

Substituting (13) and (15) in (6) and integrating by parts with respect to x, one finds:

$$\bar{I} = \pi^4 (m_1 + m_2)^2 \sum_{k \geq 2} \frac{z^k}{k!(k+1)!} f_k(y, \lambda) \tag{16}$$

where functions $f_k(y, \lambda)$ are defined by formulas:

$$f_k(y, \lambda) = -\frac{d^k}{dy^k} y^{k+1} \frac{d^{k-1}}{dy^{k-1}} \frac{f(y, \lambda)}{y} \tag{17}$$

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$$f(y, \lambda) = \log y + \int_0^1 dx \frac{\log(xy)}{1 - xy} \frac{d}{dx} \sqrt{(1 - x)(1 - \lambda y)} \tag{18}$$

The integral in (18) can be easily evaluated and we obtain for $f(y, \lambda)$ the following explicit expression:

$$f(y, \lambda) = \log y - \frac{\sqrt{\lambda}}{y} \left[\log \left(\frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \right) \log \left(\frac{4y}{(1 - \sqrt{\lambda})^2} \right) \right.$$

$$-\mathrm{Sp} \left(-\frac{2\sqrt{\lambda}}{1 + \sqrt{\lambda}} \right) + \mathrm{Sp} \left(-\frac{4\sqrt{\lambda}}{(1 - \sqrt{\lambda})^2} \right) - \mathrm{Sp} \left(-\frac{2\sqrt{\lambda}}{1 - \sqrt{\lambda}} \right) \Big]$$

$$+ \frac{(2\lambda - y\lambda + y)}{2y\sqrt{(1 - y)(\lambda - y)}} \left[\log \left(\frac{4y}{(1 - \sqrt{\lambda})} \right) \log \left(\frac{t - c}{1 - ct} \right) + \mathrm{Sp} \left(\frac{t - 1}{t - \frac{1}{c}} \right) \right.$$

$$\left. -\mathrm{Sp} \left(\frac{t - 1}{t - c} \right) + \mathrm{Sp} \left(\frac{1 - t}{1 - \frac{t}{c}} \right) - \mathrm{Sp} \left(\frac{1 - t}{1 - ct} \right) + \mathrm{Sp} \left(\frac{1 - t^2}{1 - ct} \right) - \mathrm{Sp} \left(\frac{1 - t^2}{1 - \frac{t}{c}} \right) \right] \tag{19}$$

where $\mathrm{Sp}(x)$ is the Spence function (or dilogarithm),

$$t = \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}}, \quad c = \frac{1}{y(1 - \lambda)} \left(2\lambda - y(\lambda + 1) + 2\sqrt{\lambda(1 - y)(\lambda - y)} \right) \tag{20}$$

Formulas (16), (17), and (19) give explicit expressions for coefficients at powers of p^2 in Taylor expansion of "sunset" diagram. But formula (19) seems too complicated. Fortunately, in almost all applications $m_1 = m_2$ (that is, $\lambda = 0$), and just in this case formula (19) considerably simplifies:

$$f(y, 0) = \log y + \frac{1}{\sqrt{y(1 - y)}} \mathrm{Cl}_2(2 \arcsin \sqrt{y}) \tag{21}$$

where $\mathrm{Cl}_2(\theta)$ is Clausen integral.

In conclusion, we will derive two series expansions for "sunset" diagram for the case of three different masses, when direct using of explicit formulas (16), (17), and (19) is difficult due to complexity of formula (19).

First, one substitutes the decomposition (8) in (6), expands integrand with respect to λ and performs integration. Then, after some elementary transformations, one obtains the following result:

$$\bar{I} = \frac{\pi^4 (p^2)^2}{(m_1 + m_2)^2} \sum_{j,k,l \geq 0} \left(\sum_{n \geq 0} \frac{(k+l+n+1)! \, (k+l+n+2)! \, \Gamma(n+\frac{3}{2})}{n! \, (2k+l+n+5)! \, \Gamma(j+k+l+n+\frac{5}{2})} \right)$$

$$\times \frac{(j+k+l)! \, \Gamma(j-\frac{1}{2})}{l! \, j! \, \Gamma(-\frac{1}{2})} \lambda^j (1-y)^l z^k \tag{22}$$

Above mentioned operations are correct if condition (13) is valid for all $\sigma^2 \geq (m_1 + m_2)^2$. This means, that series (22) converges, at least, at region (2).

The sum over n can be evaluated in terms of finite sums. But the corresponding expression is too cumbersome to be useful. So, if it is desirably to have closed expression for coefficients at $\lambda^j y^l z^k$, it is preferable to use another series expansion for $\bar{I}$, namely:

$$\bar{I} = \pi^4 (m_1 + m_2)^2 \sum_{j \geq 0, k \geq 2} \frac{z^k \lambda^j}{2j!} \Gamma(j + \frac{1}{2}) \left\{ \frac{(j+k-2)!}{k(k+1) \, \Gamma(j+k+\frac{1}{2})} \right.$$

$$\left. + \sum_{l \geq 0} \frac{(k+l)! \, (k+l+1)! \, (j+k+l-1)!}{k! \, (k+1)! \, l! \, (l+1)! \, \Gamma(j+k+l+\frac{3}{2})} [h_{jkl} + \log y] \, y^{l+1} \right\} \tag{23}$$

where

$$h_{jkl} = \psi(j+k+l) + \psi(k+l+1) + \psi(k+l+2)$$

$$-\psi(j+k+l+\frac{3}{2}) - \psi(l+1) - \psi(l+2)$$

In order to prove (23), it is sufficient to write function $_2F_1$ in (13) as power series with respect to y and $\log y$, to substitute this series in (6), and to perform the integration.

These operations again are correct, if condition (11) is valid for all $\sigma^2 \geq (m_1 + m_2)^2$. So the region of convergence of series (23) is not less than one defines by formula (2).

Large momentum expansion for "sunset" diagram also may be derived from integral representation (3). But in this case results of [3] seem quite exhaustive and so further investigations of this problem are unnecessarily.

Author is indebted to A.I.Davydychev and B.A.Smirnov for interesting discussions and comments.

References

[1] A.I.Davydychev, J.B.Tausk, Nucl. Phys., **B397**, 123 (1993)

[2] A.I.Davydychev, V.A.Smirnov, J.B.Tausk, Nucl. Phys., **B410**, 325 (1993)

[3] F.A.Berends, M.Buza, M.Böhm, R.Scharf, Leiden preprint INLO-PUB-17-93 (1994), Z. Phys. C, to be published

[4] F.A.Berends, J.B.Tausk, Nucl.Phys., **B421**, 456 (1994)

[5] A.Ghinculov, J.J. van der Bij, Freiburg preprint IHEP 94/05 (1994)

[6] E.Mendels, Nuovo Cimento **A45**, 87 (1978)

[7] F.A.Lunev, Phys. Rev. D, to be published (paper DMR513)

[8] D.J.Broadhurst, Z. Phys., **C47**, 115 (1990)

[9] D.J.Broadhurst, J.Fleisher, O.V.Tarasov, Z.Phys., **C60**, 287 (1993)

[10] A.I.Davydychev, J. Math. Phys., **33**, 358 (1992)

[11] A.Erdelyi, F.Oberhettinger, F.G.Tricomi,*Higher Transcendental Functions (Bateman manuscript project), vol.1*, (Mc Graw-Hill, NY, 1953)